The characterization of Hermitian surfaces by the number of points

Masaaki Homma *
Department of Mathematics and Physics
Kanagawa University
Hiratsuka 259-1293, Japan
homma@kanagawa-u.ac.jp

Seon Jeong Kim †
Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
skim@gnu.kr

Abstract

The nonsingular Hermitian surface of degree $\sqrt{q}+1$ is characterized by its number of $\mathbb{F}_q$-points among the irreducible surfaces over $\mathbb{F}_q$ of degree $\sqrt{q}+1$ in the projective 3-space.
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1 Introduction

Hermitian varieties are known as ones having particular properties over finite fields. Throughout this paper, except Section 3, q is an even power of a prime number p. A Hermitian variety over $\mathbb{F}_q$ is a hypersurface in $\mathbb{P}^n$ defined by

$$(X_0^{\sqrt{q}}, \dots, X_n^{\sqrt{q}})A\,{}^t(X_0, \dots, X_n) = 0, \tag{1}$$

where A is a square matrix of order $n+1$ whose entries are in $\mathbb{F}_q$ with the property ${}^tA = A^{(\sqrt{q})}$; here $A^{(\sqrt{q})}$ means taking entry-wise the $\sqrt{q}$-th power, and ${}^tA$ is the transposed matrix of A. We refer this kind of polynomial as a Hermitian polynomial over $\mathbb{F}_q$. For a homogeneous polynomial F of degree $\sqrt{q}+1$ whose coefficients are

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in $\mathbb{F}_q$, the hypersurface given by $F = 0$ is Hermitian if and only if there is an element $\rho \in \mathbb{F}_q^*$ such that ρF is a Hermitian polynomial. The family of Hermitian polynomials over $\mathbb{F}_q$ forms an $\mathbb{F}_{\sqrt{q}}$ vector space. It is obvious that the Hermitian polynomial (1) defines a nonsingular Hermitian variety if and only if $\det A \neq 0$. By the standard argument, a nonsingular Hermitian variety is projectively equivalent to the variety

$$H_{n-1} : X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + \cdots + X_n^{\sqrt{q}+1} = 0 \tag{2}$$

over $\mathbb{F}_q$.

The number $N_q(H_{n-1})$ of $\mathbb{F}_q$-points of H_{n-1} is

$$N_q(H_{n-1}) = (\sqrt{q}^{n+1} + (-1)^n)(\sqrt{q}^n + (-1)^{n-1})/(q-1), \tag{3}$$

which is due to [1].

This number is remarkable in the following sense. The Weil conjecture [13] established by Deligne [2] implies that the number N of $\mathbb{F}_q$-points of a nonsingular hypersurface of degree d in $\mathbb{P}^n$ defined over $\mathbb{F}_q$ is bounded by

$$N \leq \frac{q^n - 1}{q - 1} + \frac{d-1}{d}\left((d-1)^n - (-1)^n\right)\sqrt{q}^{n-1}. \tag{4}$$

Furthermore, if equality holds in (4) for a certain nonsingular hypersurface of degree d, then $d \leq \sqrt{q}+1$. As for this additional claim, see [11, Corollary 2.2] or [8, Corollary 4.3]. The number $N_q(H_{n-1})$ achieves the equality in (4) for $d = \sqrt{q} + 1$.

On the other hand, Rück and Stichtenoth's characterization of nonsingular Hermitian curves [9] implies the following fact.

Theorem 1.1 (Rück-Stichtenoth) *An absolutely irreducible plane curve of degree $\sqrt{q} + 1$ over $\mathbb{F}_q$ which has $\sqrt{q}^3 + 1$ points over $\mathbb{F}_q$ is a nonsingular Hermitian curve.*

We explain a little more detail of this assertion in the next section.

The purpose of this note is to show a similar fact for surfaces in $\mathbb{P}^3$.

Theorem 1.2 *An absolutely irreducible surface over $\mathbb{F}_q$ of degree $\sqrt{q}+1$ in $\mathbb{P}^3$ which has $(q+1)(\sqrt{q}^3 + 1)$ points over $\mathbb{F}_q$ is a nonsingular Hermitian surface.*

Notation When X is an algebraic set in $\mathbb{P}^n$ defined by equations over $\mathbb{F}_q$, the set of $\mathbb{F}_q$-points in X is denoted by $X(\mathbb{F}_q)$, and the cardinality of $X(\mathbb{F}_q)$ by $N_q(X)$.

2 Hermitian curves

For a plane curve C over $\mathbb{F}_q$, we proved a simple bound for $N_q(C)$ in a series of papers [5, 6, 7], which had been originally conjectured by Sziklai [12].

Lemma 2.1 (Sziklai bound) *Let d be an integer with $2 \leq d \leq q+2$, and C a curve of degree d in $\mathbb{P}^2$ defined over $\mathbb{F}_q$ without $\mathbb{F}_q$-linear components. Then*

$$N_q(C) \leq (d-1)q+1 \tag{5}$$

except for curves over $\mathbb{F}_4$ which are projectively equivalent to the curve defined by

$$(X_0+X_1+X_2)^4+(X_0X_1+X_1X_2+X_2X_0)^2+X_0X_1X_2(X_0+X_1+X_2)=0.$$

For the exceptional curve above, the number of its $\mathbb{F}_4$-points is 14.

The number $N_q(H_1)$ attains the equality of the Sziklai bound too. In the next proposition, we consider a slightly larger family of plane curves than the family of irreducible ones, that is to say, the family of plane curves over $\mathbb{F}_q$ without $\mathbb{F}_q$-linear components.

Proposition 2.2 *Let C be a plane curve of degree $\sqrt{q}+1$ over $\mathbb{F}_q$ without $\mathbb{F}_q$-linear components. If $N_q(C)=\sqrt{q}^3+1$, then C is a nonsingular Hermitian curve over $\mathbb{F}_q$.*

Proof. From [6, Propositions 2.1, 2.2 and 2.3], C must be absolutely irreducible and each $\mathbb{F}_q$-point of C is nonsingular, because equality holds in (5) for C. If C has singular points, all of which are not $\mathbb{F}_q$-points, then the normalization $\tilde{C}$ of C is also defined over $\mathbb{F}_q$ and $N_q(\tilde{C})=N_q(C)$. From the Hasse-Weil bound,

$$N_q(\tilde{C}) \leq 1+q+2\tilde{g}\sqrt{q} < \sqrt{q}^3+1$$

because the genus $\tilde{g}$ of $\tilde{C}$ is less than the arithmetic genus of C, that is, $\tilde{g} < \sqrt{q}(\sqrt{q}-1)/2$. Hence C is a nonsingular curve of genus $\sqrt{q}(\sqrt{q}-1)/2$. Therefore C must be a nonsingular Hermitian curve by the main theorem of [9]. □

Remark 2.3 In the previous proposition, the assumption that C has no $\mathbb{F}_q$-linear components is actually necessary. We construct a plane curve C which has $\mathbb{F}_q$-linear components such that $N_q(C)=\sqrt{q}^3+1$. Let D be an irreducible conic over $\mathbb{F}_q$ in $\mathbb{P}^2$. Let $P_0 \in D(\mathbb{F}_q)$ and L_0 the tangent line to D at P_0. Choose $P_1 \in L_0(\mathbb{F}_q) \setminus \{P_0\}$. If the characteristic p is 2, P_1 should be taken in $L_0(\mathbb{F}_q) \setminus \{P_0$, the nucleus of $D\}$. Then there are $\frac{q-1}{2}$ $\mathbb{F}_q$-lines passing through P_1 that do not meet with any point of $D(\mathbb{F}_q)$ if $p \neq 2$; or $\frac{q}{2}$ if $p=2$. Since $\sqrt{q}-2 < \frac{q-1}{2}$, we can take $\sqrt{q}-2$ such lines, say $L_1, \ldots, L_{\sqrt{q}-2}$. Then the number of $\mathbb{F}_q$-points of the curve $C=D+\sum_{i=0}^{\sqrt{q}-2} L_i$ is $\sqrt{q}^3+1$. In fact,

$$\begin{aligned} N_q(C) &= {}^{\#}(D(\mathbb{F}_q)\setminus\{P_0\})+{}^{\#}(L_0(\mathbb{F}_q)\setminus\{P_0,P_1\})+\sum_{i=1}^{\sqrt{q}-2}(L_i(\mathbb{F}_q)\setminus\{P_1\})+2 \\ &= q+(q-1)+(\sqrt{q}-2)q+2=\sqrt{q}^3+1. \end{aligned}$$

We close this section with a remark on plane curves that are used in the proof of the main theorem.

Lemma 2.4 (Segre) *Let d be an integer with $1 \leq d \leq q+1$, and C be a curve of degree d in $\mathbb{P}^2$ defined over $\mathbb{F}_q$, which may have $\mathbb{F}_q$-linear components. Then $N_q(C) \leq dq+1$, and equality holds if and only if C is a pencil of d $\mathbb{F}_q$-lines.*

Proof. See Segre [10, II, §6 Observation IV] or Homma-Kim [5, Remark 1.2]. □

3 An elementary bound

In this section, q is simply a power of p, that is, it need not be an even power of p.

In [8], we established an upper bound for $N_q(X)$ of a hypersurface X over $\mathbb{F}_q$ without $\mathbb{F}_q$-linear components, particularly a surface in $\mathbb{P}^3$ without $\mathbb{F}_q$-plane components.

Theorem 3.1 *Let S be a surface of degree d in $\mathbb{P}^3$ defined over $\mathbb{F}_q$ without $\mathbb{F}_q$-plane components. Then*

$$N_q(S) \leq (d-1)q^2 + dq + 1. \tag{6}$$

We refer the bound (6) as the elementary bound. When $d = \sqrt{q}+1$, the Weil-Deligne bound (4) for $n=3$ agrees with the elementary bound. In this section, we investigate the geometry of a surface in $\mathbb{P}^3$ whose number of $\mathbb{F}_q$-points achieves the bound (6). There are at least two examples other than the Hermitian surface each of which attains the equality in (6).

From now on, we keep the following setup until the end of this section.

Setup 3.2 Let d be an integer with $2 \leq d \leq q+1$. Let S be a surface of degree d in $\mathbb{P}^3$ defined over $\mathbb{F}_q$ without $\mathbb{F}_q$-plane components. Furthermore we suppose that $N_q(S)$ achieves the equality in (6).

Lemma 3.3 *The surface S contains an $\mathbb{F}_q$-line.*

Proof. Suppose S does not contain any $\mathbb{F}_q$-lines. Let H be any $\mathbb{F}_q$-plane in $\mathbb{P}^3$. Then $S \cap H$ is a plane curve of degree d over $\mathbb{F}_q$ in $H = \mathbb{P}^2$, and has no $\mathbb{F}_q$-line as a component. Hence

$$N_q(S \cap H) \leq \begin{cases} (d-1)q+1 & \text{if } (d,q) \neq (4,4) \\ 14 & \text{if } (d,q) = (4,4) \end{cases} \tag{7}$$

by Lemma 2.1. In a term of [4], defining the s-degree δ of $S(\mathbb{F}_q)$ by

$$\delta = \max\{N_q(S \cap H) \mid H \text{ is an } \mathbb{F}_q\text{-plane}\},$$

we have

$$N_q(S) \leq (\delta-1)q + 1 + \left\lfloor \frac{\delta-1}{q+1} \right\rfloor$$

by [4, Proposition 2.2]. Hence if $(d,q) \neq (4,4)$, then

$$
\begin{aligned}
N_q(S) &\leq (d-1)q^2 + 1 + \left\lfloor \frac{(d-1)q}{q+1} \right\rfloor \\
&= (d-1)q^2 + 1 + \left\lfloor \frac{(d-2)(q+1) + q + 2 - d}{q+1} \right\rfloor \\
&= (d-1)(q^2+1)\,;
\end{aligned}
$$

and if $(d,q) = (4,4)$, then $N_q(S) \leq 55$. In either case, $N_q(S)$ can't be $(d-1)q^2+dq+1$. $\square$

Definition 3.4 Let $l_1, \ldots, l_d$ be $\mathbb{F}_q$-lines in $\mathbb{P}^3$ with $d \geq 2$. The union of those d-lines $Z = \cup_{i=1}^d l_i$ is called a planar $\mathbb{F}_q$-pencil of degree d if those d-lines lie on a plane simultaneously and $l_1 \cap \cdots \cap l_d \neq \emptyset$. The $\mathbb{F}_q$-point $\{v_Z\} = l_1 \cap \cdots \cap l_d$ is called the vertex of Z.

Notation 3.5 For an $\mathbb{F}_q$-line l of $\mathbb{P}^3$, the set of $\mathbb{F}_q$-planes containing the line l is denoted by $\check{l}(\mathbb{F}_q)$.

Lemma 3.6 *Let l be an $\mathbb{F}_q$-line on the surface S.*

(i) *If an $\mathbb{F}_q$-plane H contains l, then $S \cap H$ is a planar $\mathbb{F}_q$-pencil of degree d.*

(ii) *The map $\check{l}(\mathbb{F}_q) \ni H \mapsto v_{S\cap H} \in l(\mathbb{F}_q)$ is bijective.*

Proof. (i) Since S has no $\mathbb{F}_q$-plane components, $S \cap H$ is a plane curve of degree d in H, and $N_q(S \cap H) \leq dq+1$ by Lemma 2.4. Counting the cardinality of $S(\mathbb{F}_q)$ by the decomposition

$$
S(\mathbb{F}_q) = \cup_{H \in \check{l}(\mathbb{F}_q)} \left((S \cap H)(\mathbb{F}_q) \setminus l(\mathbb{F}_q)\right) \bigcup l(\mathbb{F}_q),
$$

we have

$$
\begin{aligned}
(d-1)q^2 + dq + 1 = N_q(S) &\leq (q+1)(dq+1-(q+1)) + (q+1) \\
&= (d-1)q^2 + dq + 1.
\end{aligned}
$$

Hence $N_q(S \cap H) = dq+1$. So $S \cap H$ is a planar $\mathbb{F}_q$-pencil of degree d by the latter part of Lemma 2.4.

(ii) Let $\check{l}(\mathbb{F}_q) = \{H_1, H_2, \ldots, H_{q+1}\}$. Since $S \cap H_i$ is a planar $\mathbb{F}_q$-pencil and has the line l as a component, we may set notations as

$$
S \cap H_i = l \cup l_{i,1} \cup \cdots \cup l_{i,d-1}
$$

and

$$
v_i = v_{S \cap H_i}.
$$

It is obvious that $v_i \in l(\mathbb{F}_q)$. Since $\check{l}(\mathbb{F}_q)$ and $l(\mathbb{F}_q)$ have the same cardinality, it is enough to show that the map $\check{l}(\mathbb{F}_q) \ni H_i \mapsto v_i \in l(\mathbb{F}_q)$ is surjective.

Contrary, suppose this map is not surjective. Pick a point $Q \in l(\mathbb{F}_q) \setminus \{v_1, \ldots, v_{q+1}\}$, and choose an $\mathbb{F}_q$-plane K such that $K \ni Q$ and $K \not\supset l$. Then we face two consequences:

(α) $S \cap K$ does not contain any $\mathbb{F}_q$-line;

(β) $N_q(S \cap K) \geq (d-1)q + d$,

as we verify them below. Since $S \cap K$ is a plane curve of degree d in K, those two conditions are incompatible each other by Lemma 2.1.

The verification of (α). If $S \cap K$ contains an $\mathbb{F}_q$-line, then it must be a planar $\mathbb{F}_q$-pencil by (i). Since Q is a point of S, there is an $\mathbb{F}_q$-line m passing through Q among d lines of $S \cap K$. Hence l and m spans an $\mathbb{F}_q$-plane which is one of the H_i's, say H_i. Then $v_i \in l \cap m = \{Q\}$, which contradicts to the choice of Q.

The verification of (β). Let $Q_{i,j}$ be the intersection point of $l_{i,j}$ with K. Then the plane containing l and $Q_{i,j}$ is H_i and the line containing v_i and $Q_{i,j}$ is $l_{i,j}$. Hence $Q_{i,j} = Q_{i',j'}$ implies $(i,j) = (i',j')$. Since

$$(S \cap K)(\mathbb{F}_q) \supset \{Q_{i,j} \mid 1 \leq i \leq q+1,\ 1 \leq j \leq d-1\} \cup \{Q\},$$

we have $N_q(S \cap K) \geq (d-1)(q+1) + 1$. □

Lemma 3.7 *Let H be an $\mathbb{F}_q$-plane such that $S \cap H$ is a planar $\mathbb{F}_q$-pencil of degree d. If an $\mathbb{F}_q$-line $l \subset S$ passes through $v_{S\cap H}$, then l is a component of $S \cap H$.*

Proof. Since $d \geq 2$, there are two distinct components l_1 and l_2 of $S \cap H$. Suppose that l is not contained in H. Then l and l_i span an $\mathbb{F}_q$-plane, say H_i. Hence $S \cap H_i$ is also a planar $\mathbb{F}_q$-pencil of degree d by Lemma 3.6 (i). By the construction of H_1 and H_2,

$$v_{S\cap H_1} = l \cap l_1 = v_{S\cap H} = l \cap l_2 = v_{S\cap H_2},$$

which contradicts (ii) of Lemma 3.6. Hence l is contained in H, and hence it is a component of $S \cap H$. □

Corollary 3.8 *For any $\mathbb{F}_q$-point P of S, there is a unique $\mathbb{F}_q$-plane H such that $S \cap H$ is a planar $\mathbb{F}_q$-pencil of degree d with $v_{S\cap H} = P$.*

Proof. From Lemma 3.3 and Lemma 3.6 (i), there is a line l on S containing P. Hence, by (ii) of Lemma 3.6 there is a desired $\mathbb{F}_q$-plane. The uniqueness of such an $\mathbb{F}_q$-plane comes from the above lemma. □

4 A characterization of Hermitian surfaces

In this section, q is assumed to be an even power of a prime number p again, and S is a surface in $\mathbb{P}^3$ of degree $\sqrt{q}+1$ defined over $\mathbb{F}_q$ without $\mathbb{F}_q$-plane components such that $N_q(S) = (\sqrt{q}^3 + 1)(q+1)$. Note that this number $N_q(S)$ achieves both the Weil-Deligne bound (4) for $n = 3$ and the elementary bound (6).

Proposition 4.1 *With the above situation, let H be an $\mathbb{F}_q$-plane of $\mathbb{P}^3$. Then either*

(1) *$S \cap H$ is a planar $\mathbb{F}_q$-pencil of degree $\sqrt{q}+1$, or*

(2) *$S \cap H$ is a nonsingular Hermitian curve of degree $\sqrt{q}+1$ over $\mathbb{F}_q$.*

Furthermore, let ν_1 denote the number of $\mathbb{F}_q$-planes having the property (1) *above, and ν_2 the property* (2)*. Then*

$$\begin{cases} \nu_1 &= N_q(S) = (\sqrt{q}^3+1)(q+1) \\ \nu_2 &= \sqrt{q}^3(q+1)(\sqrt{q}-1). \end{cases}$$

Proof. Let $\check{\mathbb{P}}^3(\mathbb{F}_q)$ denote the set of $\mathbb{F}_q$-planes of $\mathbb{P}^3$, and $\mathcal{P}_1$ the set of $\mathbb{F}_q$-planes having the property (1). The map $\mathcal{P}_1 \ni H \mapsto v_{S\cap H} \in S(\mathbb{F}_q)$ is bijective by Corollary 3.8. Hence $\nu_1 = N_q(S)$. Let $\mathcal{P}_2 = \check{\mathbb{P}}^3(\mathbb{F}_q) \setminus \mathcal{P}_1$. Note that if $H \in \mathcal{P}_2$, then $S \cap H$ has no $\mathbb{F}_q$-liner components by Lemma 3.6 (i). Consider the correspondence

$$\mathcal{A} = \{(P, H) \mid P \in H\} \subset S(\mathbb{F}_q) \times \check{\mathbb{P}}^3(\mathbb{F}_q)$$

with two projections $\pi_1 : \mathcal{A} \to S(\mathbb{F}_q)$ and $\pi_2 : \mathcal{A} \to \check{\mathbb{P}}^3(\mathbb{F}_q)$. Counting the cardinality of $\mathcal{A}$ by π_1, we have

$$^{\#}\mathcal{A} = N_q(S) \cdot (q^2+q+1).$$

If $H \in \mathcal{P}_1$, then $^{\#}\pi_2^{-1}(H) = N_q(S \cap H) = \sqrt{q}^3 + q + 1$, and if $H \in \mathcal{P}_2$, then $^{\#}\pi_2^{-1}(H) = N_q(S \cap H) \leq \sqrt{q}^3 + 1$ by Lemma 2.1. Hence, counting the cardinality of $\mathcal{A}$ by π_2, we have

$$\begin{aligned} ^{\#}\mathcal{A} &\leq N_q(S) \cdot (\sqrt{q}^3 + q + 1) + \left(\frac{q^4-1}{q-1} - N_q(S)\right)(\sqrt{q}^3+1) \\ &= N_q(S) \cdot (q^2+q+1). \end{aligned}$$

Therefore, if $H \in \mathcal{P}_2$, then $N_q(S \cap H) = \sqrt{q}^3 + 1$. Furthermore, when $H \in \mathcal{P}_2$, since $S \cap H$ has no $\mathbb{F}_q$-line as a component, $S \cap H$ is a nonsingular Hermitian curve by Proposition 2.2. $\square$

Now we come to the main theorem. From the above proposition, we know that for any $\mathbb{F}_q$-line l, the cardinality of $S(\mathbb{F}_q) \cap l$ is either 1 or $\sqrt{q}+1$ or $q+1$. So we may apply the characterization of nonsingular Hermitian surfaces by Hirschfeld [3, Theorem 19.5.12], though we give a straightforward proof because our starting point is a surface given by an equation. In our proof, the hyperplane defined by an equation $h = 0$ is sometimes denoted by $\{h = 0\}$; and also $\mathbb{F}_q \setminus \{0\}$ by $\mathbb{F}_q^*$.

Theorem 4.2 *Let S be a surface in $\mathbb{P}^3$ defined over $\mathbb{F}_q$ without $\mathbb{F}_q$-plane components. If the degree of S is $\sqrt{q}+1$ and $N_q(S) = (\sqrt{q}^3+1)(q+1)$, then S is a nonsingular Hermitian surface over $\mathbb{F}_q$.*

Proof. From Proposition 4.1, there is an $\mathbb{F}_q$-plane $H_\infty \subset \mathbb{P}^3$ such that $S \cap H_\infty$ is a nonsingular Hermitian curve. Choose a system of homogeneous coordinates X_0, X_1, X_2, X_3 of $\mathbb{P}^3$ such that

(i) H_∞ is given by $X_0 = 0$; and

(ii) the plane curve $S \cap H_\infty$ in $H_\infty = \mathbb{P}^2$ is given by

$$\bar{X}_1\bar{X}_2^{\sqrt{q}} + \bar{X}_1^{\sqrt{q}}\bar{X}_2 + \bar{X}_3^{\sqrt{q}+1} = 0,$$

where $\bar{X}_1, \bar{X}_2, \bar{X}_3$ are coordinates on H_∞ induced by X_1, X_2, X_3 respectively.

Let $P_1 = (0,0,1,0)$ and $P_2 = (0,1,0,0)$, both of which are points on $S \cap H_\infty$. For $\alpha = 1, 2$, the tangent line L_α at P_α to $S \cap H_\infty$ in $H_\infty = \mathbb{P}^2$ is given by $\bar{X}_\alpha = 0$. It is easy to see that $S \cap L_\alpha = \{P_\alpha\}$. Hence for an $\mathbb{F}_q$-plane $H \supset L_\alpha$,

$$\#\left(S(\mathbb{F}_q) \cap (H \setminus L_\alpha)\right) = \begin{cases} \sqrt{q}^3 + 1 - 1 & \text{if } S \cap H \text{ is Hermitian} \\ \sqrt{q}^3 + q + 1 - 1 & \text{if } S \cap H \text{ is an } \mathbb{F}_q\text{-pencil.} \end{cases}$$

Counting the number $N_q(S) - 1$ by using all $\mathbb{F}_q$-planes containing L_α, we know that there is a unique plane $H_{\alpha,0} \supset L_\alpha$ such that $S \cap H_{\alpha,0}$ is a planar $\mathbb{F}_q$-pencil of degree $\sqrt{q}+1$; and $S \cap H_{\alpha,\lambda}$ is a nonsingular Hermitian curve for other plane $H_{\alpha,\lambda} \supset L_\alpha$. By changing coordinates of type

$$\begin{cases} X_1 & \mapsto \quad X_1 + aX_0 \\ X_2 & \mapsto \quad X_2 + bX_0 \end{cases}$$

if necessary, we may suppose that $H_{\alpha,0}$ is defined by $X_\alpha = 0$ for $\alpha = 1$ and 2 respectively. But the situation on H_∞ never change.

Summing up, S is defined by

$$F(X_0, \ldots, X_3) = X_0 f(X_0, \ldots, X_3) + h(X_1, X_2, X_3) = 0$$

where

$$h(X_1, X_2, X_3) = X_1X_2^{\sqrt{q}} + X_1^{\sqrt{q}}X_2 + X_3^{\sqrt{q}+1}$$

and $\deg f = \sqrt{q}$.

Since $S \cap H_{1,0} = S \cap \{X_1 = 0\}$ is a planar $\mathbb{F}_q$-pencil with the vertex $P_1 = (0,0,1,0)$,

$$F(X_0, 0, X_2, X_3) = X_0 f(X_0, 0, X_2, X_3) + X_3^{\sqrt{q}+1}$$

does not contain X_2, because this polynomial must have the form $c\prod_j(X_0 + \gamma_j X_3)$, with $c \in \mathbb{F}_q^*$. Hence, in $F(X_0, X_1, X_2, X_3)$, any monomial containing X_2 also contains X_1. By the same argument on $S \cap H_{2,0} = S \cap \{X_2 = 0\}$, any monomial containing X_1 also contains X_2. Therefore $f(X_0, X_1, X_2, X_3)$ is written as

$$f(X_0, X_1, X_2, X_3) = g_1(X_0, X_3) + g_2(X_0, \ldots, X_3)X_1X_2,$$

where $\deg g_1 = \sqrt{q}$ and $\deg g_2 = \sqrt{q} - 2$.

For the plane $H_{1,\lambda} = \{X_0 = \lambda X_1\}$ ($\lambda \in \mathbb{F}_q^*$), since $S \cap \{X_0 = \lambda X_1\}$ is a Hermitian curve,

$$\begin{aligned} &\rho F(\lambda X_1, X_1, X_2, X_3) = \\ &\qquad \rho\left(\lambda X_1(g_1(\lambda X_1, X_3) + g_2(\lambda X_1, X_1, X_2, X_3)X_1X_2) + h(X_1, X_2, X_3)\right) \end{aligned} \tag{8}$$

is a Hermitian polynomial for some $\rho \in \mathbb{F}_q^*$. Since $X_3^{\sqrt{q}+1}$ appears only in $h(X_1, X_2, X_3)$, the constant ρ must be an element of $\mathbb{F}_{\sqrt{q}}^*$. Hence (8) is a Hermitian polynomial even if $\rho = 1$. Since h itself Hermitian,

$$\lambda g_1(\lambda X_1, X_3)X_1 + \lambda g_2(\lambda X_1, X_1, X_2, X_3)X_1^2 X_2$$

must be Hermitian.

Any monomial in $g_2(\lambda X_1, X_1, X_2, X_3)X_1^2 X_2$ never appear in $g_1(\lambda X_1, X_3)X_1$. So $g_2(\lambda X_1, X_1, X_2, X_3))X_1^2 X_2$ contains the monomials only of types $X_i^{\sqrt{q}+1}$ or $X_i^{\sqrt{q}} X_j$. Hence

$$g_2(\lambda X_1, X_1, X_2, X_3)X_1^2 X_2 = \mu X_1^{\sqrt{q}} X_2 \ (\mu \in \mathbb{F}_q).$$

But the monomial $X_1 X_2^{\sqrt{q}+1}$ can't appear in

$$g_1(\lambda X_1, X_3)X_1 + g_2(\lambda X_1, X_1, X_2, X_3)X_1^2 X_2.$$

Hence $g_2(\lambda X_1, X_1, X_2, X_3) = 0$, that is, $X_0 - \lambda X_1$ is a factor of $g_2(X_0, X_1, X_2, X_3)$ for any $\lambda \in \mathbb{F}_q^*$. But $\deg g_2 = \sqrt{q} - 2\,(< q-1)$, it is impossible. Therefore $g_2(X_0, X_1, X_2, X_3) = 0$ and $\lambda g_1(\lambda X_1, X_3)X_1$ is Hermitian.

Let

$$g_1(X_0, X_3) = \sum_{i=0}^{\sqrt{q}} a_i X_0^i X_3^{\sqrt{q}-i}.$$

Then

$$\lambda g_1(\lambda X_1, X_3)X_1 = \sum_{i=0}^{\sqrt{q}} a_i \lambda^{i+1} X_1^{i+1} X_3^{\sqrt{q}-i},$$

which is Hermitian. Hence $a_i = 0$ for $i \neq 0$, $\sqrt{q} - 1$, $\sqrt{q}$; and also $a_0^{\sqrt{q}} = a_{\sqrt{q}-1}$ and $a_{\sqrt{q}} \in \mathbb{F}_{\sqrt{q}}$, that is,

$$g_1(X_0, X_3) = a_0 X_3^{\sqrt{q}} + a_0^{\sqrt{q}} X_0^{\sqrt{q}-1} X_3 + a_{\sqrt{q}} X_0^{\sqrt{q}} \text{ with } a_{\sqrt{q}} \in \mathbb{F}_{\sqrt{q}}.$$

Hence

$$F(X_0, \ldots, X_3) = a_0 X_0 X_3^{\sqrt{q}} + a_0^{\sqrt{q}} X_0^{\sqrt{q}} X_3 + a_{\sqrt{q}} X_0^{\sqrt{q}+1} + h(X_1, X_2, X_3),$$

which is Hermitian. If $a_{\sqrt{q}} = 0$, then the surface is a cone of a nonsingular Hermitian curve with vertex $(1, 0, 0, 0)$, and then $N_q(S) = (\sqrt{q}^3 + 1)q + 1$, which is not the given number. So $a_{\sqrt{q}} \in \mathbb{F}_{\sqrt{q}}^*$, and hence S is a nonsingular Hermitian surface. □

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